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We propose a scheme for measuring Berry phase in the vibrational degree of freedom of a trapped ion. Starting from the ion in a vibrational coherent state we show how to reverse the sign of its amplitude by using a purely geometric phase. This can then be detected through internal degrees of freedom of the ion.

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When the Hamiltonian of a quantum system is varied adiabatically in a cyclic fashion, the state of the system acquires a geometrical phase in addition to the usual dynamical phase. This surprising effect, discovered by Berry [1], (and generalized to arbitrary cyclic evolutions by Aharonov and Anadan [2]) has been extensively tested for two level systems [3] and attracted interest from a wide variety of fields [4,5]. However, Berry's phase has not been experimentally measured for harmonic oscillators, though some theoretical calculations exist [6–8]. The reason for this might be the fact that for the simplest case, namely for adiabatic displacement of an oscillator state in phase space, the Berry phase is independent of the state [6], and thereby undetectable. However, when a squeezing Hamiltonian is switched on, and the squeezing parameter is varied slowly and cyclically, there would be a detectable Berry phase [6,7]. For an initial Fock state $|n\rangle$ which undergoes squeezing, the Berry phase after a cycle is $-(n+1/2)$ times the classical Hannay angle [6,7]. The quantized states of the electromagnetic field would have been a natural candidate to test this kind of phase, but squeezed states, being nonclassical, are not stable enough for an adiabatic evolution. However, the vibrational mode of a trapped ion has been a fertile ground for the preparation of long lived nonclassical states of a harmonic oscillator [9–12]. In this letter, we derive a Berry phase formula for a certain adiabatic evolution of a joint state of the internal levels of a trapped ion and its vibrational motion. Despite being the phase gained by a joint state, its value is fundamentally dependent on the harmonic oscillator nature of the vibrational mode. We propose a scheme a detect this phase which is feasible with current technology.

Consider the Hamiltonian

$$H = H_a + H_b, \tag{1}$$

where

$$H_a = g_a e^{i\phi}|e\rangle\langle g|a + h.c., \tag{2}$$

$$H_b = g_b|e\rangle\langle g|a^\dagger + h.c. \ . \tag{3}$$

In the above, $|e\rangle$ and $|g\rangle$ are two states of a qubit, a and $a^\dagger$ are the creation and annihilation operators of a harmonic oscillator, g_a and g_b are unequal positive interaction strengths (say $g_a > g_b$) and ϕ is an arbitrary phase factor. The physical motivation of choosing this Hamiltonian will be clarified later. If the phase ϕ is slowly varied over a complete loop (so that the adiabatic approximation holds true), there will be a nontrivial Berry phase acquired by an eigenstate of the Hamiltonian H. We now proceed to calculate this. We transform the Hamiltonian as

$$H^{'} = S(\epsilon)^\dagger H S(\epsilon) \tag{4}$$

where

$$S(\epsilon)^\dagger a S(\epsilon) = a\cosh(r) - a^\dagger \sinh(r) e^{i\theta} \tag{5}$$

is a squeezing transformation with squeezing parameter $\epsilon = re^{i\theta}$. If we chose the squeezing strength $r = \tanh^{-1} g_b/g_a$ and the squeezing phase $\theta = -\phi$, the transformed Hamiltonian will be

$$H^{'} = \Omega(|e\rangle\langle g|a + a^\dagger|g\rangle\langle e|), \tag{6}$$

where $\Omega = g_a\cosh(r) - g_b\sinh(r)$. $H^{'}$ is the well known Jaynes-Cummings Hamiltonian [13]. The eigenstates of this Hamiltonian is

$$|\Psi_n^\pm\rangle = \frac{1}{\sqrt{2}}(|g\rangle|n+1\rangle \pm |e\rangle|n\rangle). \tag{7}$$

This implies that the eigenstates of our original Hamiltonian H are

$$|\Psi_n^\pm(\epsilon)\rangle = \frac{S(\epsilon)}{\sqrt{2}}(|g\rangle|n+1\rangle \pm |e\rangle|n\rangle). \tag{8}$$

Now we can proceed to calculate the Berry phase from the instantaneous eigenstates $|\Psi_n^\pm(\epsilon)\rangle$ of H.

The expression for the Berry phase for an adiabatic cyclic evolution of a Hamiltonian $H(R)$ in parameter space R is given in terms of the instantaneous eigenstates $|n, \mathbf{R}\rangle$ of the Hamiltonian as

$$\gamma_n = i\int_c d\mathbf{R}\langle n, \mathbf{R}|\nabla_R|n, \mathbf{R}\rangle \tag{9}$$

. Using Eq.(8) in the above equation we get

$$\gamma_n = i\int_c d\epsilon \langle \Psi_n^{\pm}|S(\epsilon)^{\dagger}\nabla_{\epsilon}S(\epsilon)|\Psi_n^{\pm}\rangle$$
$$= \frac{i}{2}\int_c d\epsilon \langle n+1|S(\epsilon)^{\dagger}\nabla_{\epsilon}S(\epsilon)|n+1\rangle$$
$$+ \frac{i}{2}\int_c d\epsilon \langle n|S(\epsilon)^{\dagger}\nabla_{\epsilon}S(\epsilon)|n\rangle. \quad (10)$$

If the modulus r of the parameter ϵ is kept constant throughout the evolution, then using the expression for $\langle n|S(\epsilon)^{\dagger}\nabla_{\epsilon}S(\epsilon)|n\rangle$ from Ref. [6] we get

$$\gamma_n = -2\pi(n+1)\sinh^2 r. \quad (11)$$

Note that the Berry phase γ_n is same for both the states $|\Psi_n^+(\epsilon)\rangle$ and $|\Psi_n^-(\epsilon)\rangle$. However, the dynamical phase $\beta_n^{\pm}$ is exactly opposite for the eigenstates $|\Psi_n^+(\epsilon)\rangle$ and $|\Psi_n^-(\epsilon)\rangle$ and is given by

$$\beta_n^{\pm} = -\int_c d\epsilon \langle \Psi_n^{\pm}|S(\epsilon)^{\dagger}HS(\epsilon)|\Psi_n^{\pm}\rangle$$
$$= \mp\Omega\sqrt{n+1}. \quad (12)$$

Thus one can make the dynamical phase completely vanish after two cycles by changing the state $|\Psi_n^+(\epsilon)\rangle$ to $|\Psi_n^-(\epsilon)\rangle$ or vice versa after one cycle. Under such circumstances, the only contribution to the phase of the system will be geometrical.

Let us now describe how the Hamiltonian H of Eq.(1) can be physically realized in an ion trap. Recently, ion traps have been a very active field of both theoretical [14–16] and experimental research [9–12]. Consider a single two-level atom in a harmonic trap of frequency ν. The two-level transition frequency for the atom is ω_o and $|g\rangle$ and $|e\rangle$ are the ground and excited states. The motion of the ion is modified by the interaction with two standing waves of frequency $\omega_{L1} = \omega_o - \nu$ and $\omega_{L2} = \omega_o + \nu$ which drive the transition $|g\rangle \rightarrow |e\rangle$. The Hamiltonian for this system is

$$H = H_o + H_I \quad (13)$$

where

$$H_o = \hbar\nu a^{\dagger}a + \frac{\omega_o}{2}\sigma_z, \quad (14)$$

is the Hamiltonian of the system without the laser fields and

$$H_I = -\frac{d}{\hbar}[\sigma_+ E^{(+)}(R,t) + \sigma_- E^{(-)}(R,t)] \quad (15)$$

is the interaction Hamiltonian in the dipole and optical rotating wave approximations in which $a^{\dagger}$ and a are the creation and annihilation operators for the vibrational modes, $\sigma_+ = |e\rangle\langle g|$ and $\sigma_+ = |g\rangle\langle e|$, d is the transition dipole moment and R is the position operator of the ion. In the above (Eq.(15)),

$$E^{(+)}(R,t) = E_a \sin(k_a R)e^{-i(\omega_0-\nu)t-i\phi}$$
$$+ E_b \sin(k_b R)e^{-i(\omega_0+\nu)t}. \quad (16)$$

In the above, we have assumed the phase of laser b to be zero without the loss of any generality and the relative phase of the two lasers to be ϕ. The above Hamiltonian (bichromatic excitation) can be used to generate squeezed states of motion of the ion when used in conjunction with atomic decay [15,16]. Here, however, we do not want atomic spontaneous emmission, and thus prefer to choose the $|e\rangle \rightarrow |g\rangle$ transition to be *dipole forbidden*. Hence the atomic decay rate Γ can be neglected for our purposes . Nonetheless, the above Hamiltonian H still holds (see section II.A of Ref. [16]). We can reexpress $k_i R$ as $\eta_i(a + a^{\dagger})$, where $i = a, b$. The factor $\eta = \pi a_o/\lambda$ is the Lamb-Dicke parameter where a_o is the amplitude of the ground state of the trap potential and λ the optical wavelength. In the Lamb-Dicke limit ($\eta << 1$) and assuming the ion is localized in a region much smaller than the optical wavelength (to cool to this state one could use dipole allowed transitions before switching on H). In this case the field can be expanded to the first order in η_i. Transforming into the rotating frame with $U = \exp[-i(\nu a^{\dagger}a + \omega_0\sigma_z)t]$ we get

$$H^{(1)} = U^{\dagger}HU = |e\rangle\langle g|(g_a e^{i\phi}a + g_b a^{\dagger}) + h.c.$$
$$= H_a + H_b. \quad (17)$$

where we have assumed $g_i = \eta_i\Omega_i/2 << \nu$ to get rid of the rapidly oscillating terms, Ω_i being the Rabi frequency of the ith laser. Thus we can obtain the Hamiltonian H in an ion trap by the above methods in a rotating frame.

We now describe our proposal to measure a Harmonic oscillator Berry phase in a vibrational mode of a trapped ion. It is assumed that the atom is cooled to the ground state of motion and in the internal state $|g\rangle$. First we prepare a coherent state $|\alpha\rangle$ of the vibrational mode. This can be done by shifting the centre of the trap [10] or other methods [15,11] and our scheme is completely independent of the method used to accomplish this. We require the coherent state amplitude α to be as large as possible. Next we want to achieve the evolution

$$|g\rangle|\alpha\rangle \rightarrow |g\rangle|-\alpha\rangle \quad (18)$$

by purely geometric means (i.e. *only* through the Berry phase). To this end we switch on the Hamiltonian H by switching on appropriate lasers and viewing the dynamics from the rotating frame with $U = \exp[-i(\nu a^{\dagger}a + \omega_0\sigma_z)t]$ for the moment (we will describe what happens in the laboratory frame at a later point). The initial state can be written as

$$|\alpha\rangle|g\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{2}}\sum_n \frac{\alpha^n}{\sqrt{n!}}(|\Psi_n^+\rangle + |\Psi_n^-\rangle). \quad (19)$$

Next, the relative phase ϕ of the lasers is varied very slowly and cyclically while the parameter r =

$\tanh^{-1} g_b/g_a$ is kept constant. If the time period of variation of ϕ is much larger than the Rabi oscillation time period between the vibrational mode and the internal states of the ion, then the quantum adiabatic theorem holds. As ϕ is varied, the state evolves as

$$|\Psi(\phi)\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{i\gamma_n(\phi)} (e^{i\beta_n^+(\phi)} |\Psi_n^+(\epsilon)\rangle + e^{i\beta_n^-(\phi)} |\Psi_n^-(\epsilon)\rangle), \quad (20)$$

where $\gamma_n(\phi)$ and $\beta_n^{\pm}(\phi)$ are geometric and dynamical phases respectively and $\epsilon = re^{-i\phi}$. After ϕ has completed an entire cycle (i.e. $\phi = 2\pi$), we apply a state dependent phase shift $|g\rangle \rightarrow -|g\rangle$ to the ionic state. Such a transformation can be done by applying a π pulse [12] and has to be done in a timescale much shorter that the evolution timescale of the system. This converts $|\Psi_n^+(\epsilon)\rangle \rightarrow |\Psi_n^-(\epsilon)\rangle$ and vice versa. Now we vary the phase difference ϕ again from 2π to 4π. The resulting state at the end of this second cycle is

$$|\Psi(4\pi)\rangle = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}} \sum_n \frac{\alpha^n e^{i(\gamma_n(4\pi)+\beta_n^+(2\pi)+\beta_n^-(2\pi))}}{\sqrt{n!}} (|\Psi_n^+(\epsilon)\rangle + |\Psi_n^-(\epsilon)\rangle). \quad (21)$$

As $\beta_n^+ = -\beta_n^-$, the dynamical phase completely cancels. If we, moreover, choose $\sinh^2 r = 1/4$, we have $\gamma_n(4\pi) = -\pi$. Under such circumstances, the final state would be

$$|\Psi(4\pi)\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{2}} \sum_n \frac{(-\alpha^n)}{\sqrt{n!}} (|\Psi_n^+(\epsilon)\rangle + |\Psi_n^-(\epsilon)\rangle) = |g\rangle|-\alpha\rangle. \quad (22)$$

So the detection of the Berry phase now amounts to distinguishing between the coherent states $|\alpha\rangle$ and $|-\alpha\rangle$. To this end, after switching off the adiabatic evolution, the entire state is given a negative displacement of $-\alpha$. This reduces our problem to distinguishing between $|0\rangle$ (a vibrational state with no excitation) and $|-2\alpha\rangle$. After that, the ionic internal states are allowed to interact with the vibrational mode by a Jaynes Cummings interaction. In the case of no Berry phase, the probability of finding the ion in an excited state at any time t is zero, while, in the presence of a Berry phase the same probability is given by

$$P_e = e^{-2|\alpha|^2} \sum \frac{(-2\alpha)^{2n}}{n!} \sin^2 \Omega_{n+1} t \quad (23)$$

where Ω_{n+1} is the Rabi frequency corresponding to an excitation number $n+1$. Note that the probability P_e cannot be made unity at any time and the above method is not a perfect discrimination (such a discrimination is impossible in principle as $|0\rangle$ and $|-2\alpha\rangle$ are not orthogonal).

In the laboratory (nonrotating) frame, there will be an extra geometric phase equal to $\int_c d\epsilon \langle \Psi_n^{\pm}|S(\epsilon)^{\dagger}(\nu a^{\dagger}a + \omega_0\sigma_z)S(\epsilon)|\Psi_n^{\pm}\rangle$. For our choice of $\sinh^2 r = 1/4$ this phase turns out to be $\frac{3}{2}\nu T$ where T is the time required to complete the two cycles of the phase ϕ. We can choose T in such a way that this phase becomes a multiple of 2π. This will keep the magnitude of the Berry's phase unaltered from that in the rotating frame.

Let us now point out the feasibility of our experiment with existing ion trap parameters. For adiabaticity, we require the time scale T of variation of the relative phase ϕ to be much greater than the dynamical timescale of the problem. The dynamical timescale is given by $g_i = (\Omega_i \eta_i/2)^{-1}$. For realistic ion traps we have, $\eta \sim 0.1$ and $\Omega \sim 10^3 KHz$ [17]. The dynamical timescale is then about 10^{-5} s and we just have to vary the phase of the laser at a larger time scale. Let us choose $T \sim 10^{-3}$ s. If we are using a dipole forbidden transition, this time scale is automatically much shorter than the lifetime of the excited state (which can be upto 10 s [16]). Hence, our assumption of neglecting decoherence is well justified. We also have to set $(3/2)\nu T = m2\pi$ for the dynamical phase in the nonrotating frame to vanish, where m is an integer. For standard traps, $\nu \sim 10MHz$, [15] and hence $\nu T \sim 10^4$. We can easily choose a number of such a large order to be a multiple of 2π.

It is interesting to point out that the above scheme could be performed starting with ion in the excited state. Then the whole evolution would be $|e\rangle|\alpha\rangle \rightarrow -|e\rangle|-\alpha\rangle$. Therefore it makes no difference to our scheme what the initial state of the ion is, the initial coherent state of the vibrational mode always undergoes the same evolution (i.e. the sign of the amplitude is reversed by purely geometric means). Using a similar scheme we can also create a Schrödinger cat state of the ion. One initially has to prepare the ion in a superposition of $|g\rangle$ and some other state $|r\rangle$ which is completely decoupled from the evolution due to our Hamiltonian. Then the evolution proceeds as

$$(|g\rangle + |r\rangle)|\alpha\rangle \longrightarrow |g\rangle|-\alpha\rangle + |r\rangle|\alpha\rangle \ . \quad (24)$$

The state in the above equation is a Scrödinger cat state of the ion [12].

We have shown how to observe the Berry phase of a simple harmonic oscillator using a trapped ion. We have described how to reverse the amplitude of a coherent state by purely geometric means (i.e. by using only the Berry phase). A big advantage in terms of feasibility of the experiment is that it needs only a single trapped ion. This is a meagre requirement in comparison to the technology that already exists such as the ability to entangle four ions in a trap [17]. Moreover, the existing ion trap parameters are well in range of those required for implementing our proposal and for our analysis to remain valid. We have also indicated how to use our scheme to create a Scrödinger cat state of the ion, which is an entangled state between the internal degrees of the ion and its

vibrational motion. Further interesting extensions to the geometric approach to multiple modes are also possible and will be investigated in the future.

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